Review exercise 1

1 a $8^{\frac{1}{3}}$

Use $a^{\frac{1}{m}} = \sqrt[m]{a}$, so $a^{\frac{1}{3}}$

$= \sqrt[3]{a}$

$= \sqrt[3]{8}$

$= 2$

b $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}}$ $\left(\text{Use } a^{-m} = \frac{1}{a^m}\right)$

$\left(\text{Use } a^{\frac{n}{m}} = \left(\sqrt[m]{a}\right)n\right)$

$8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2$

$8^{\frac{2}{3}} = 2^2 = 4$

$8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}}$

$= \frac{1}{4}$

2 a $125^{\frac{4}{3}}$

$a^{\frac{n}{m}} = \sqrt[m]{\left(a^n\right)}$ or $\left(\sqrt[m]{a}\right)^n$

$= \left(\sqrt[3]{125}\right)^4$

$= 5^4$

$= 625$

b $24x^2 \div 18x^{\frac{4}{3}}$

(Use $a^m \div a^n = a^{m-n}$)

$= \frac{24x^2}{18x^{\frac{4}{3}}} = \frac{4x^2}{3x^{\frac{4}{3}}}$

$= \frac{4x^{\frac{2}{3}}}{3}$ $\left(\text{because } 2 - \frac{4}{3} = \frac{2}{3}\right)$

3 a $\sqrt{80}$

Use $\sqrt{bc} = \sqrt{b}\sqrt{c}$

$= \sqrt{16} \times \sqrt{5}$

$= 4\sqrt{5}$

$(a = 4)$

b $\left(4-\sqrt{5}\right)^2 = \left(4-\sqrt{5}\right)\left(4-\sqrt{5}\right)$

$= 4\left(4-\sqrt{5}\right) - \sqrt{5}\left(4-\sqrt{5}\right)$

$= 16 - 4\sqrt{5} - 4\sqrt{5} + 5$

$= 21 - 8\sqrt{5}$

$\left(b = 21 \text{ and } c = -8\right)$

4 a $\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right)$

$= 4\left(4-\sqrt{3}\right) + \sqrt{3}\left(4-\sqrt{3}\right)$

$= 16 - 4\sqrt{3} + 4\sqrt{3} - 3$

$= 13$

b $\frac{26}{4+\sqrt{3}} \times \frac{4-\sqrt{3}}{4-\sqrt{3}} = \frac{26\left(4-\sqrt{3}\right)}{\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right)}$

$= \frac{26\left(4-\sqrt{3}\right)}{13}$

$= 8 - 2\sqrt{3}$

$\left(a = 8 \text{ and } b = -2\right)$

5 a mean $= \frac{1-\sqrt{k}+2+5\sqrt{k}+2\sqrt{k}}{3}$

$= \frac{3+6\sqrt{k}}{3}$

$= 1 + 2\sqrt{k}$

b range $= 2 + 5\sqrt{k} - (1-\sqrt{k})$

$= 1 + 6\sqrt{k}$

6 a $y^{-1} = \left(\frac{1}{25}x^4\right)^{-1}$

$= \dfrac{1}{\frac{1}{25}x^4}$

$= \dfrac{25}{x^4}$

$= 25x^{-4}$

b $5y^{\frac{1}{2}} = 5\left(\frac{1}{25}x^4\right)^{\frac{1}{2}}$

$= 5\left(\frac{1}{5}x^2\right)$

$= x^2$

7 Area $= \frac{1}{2}h(a+b)$

$= \frac{1}{2}\left(2\sqrt{2}\right)\left(3+\sqrt{2}+5+3\sqrt{2}\right)$

$= \sqrt{2}\left(8+4\sqrt{2}\right)$

$= 8\sqrt{2}+8$

The area of the trapezium is $8+8\sqrt{2}$ cm^2.

8 $\dfrac{p+q}{p-q} = \dfrac{(3-2\sqrt{2})+(2-\sqrt{2})}{(3-2\sqrt{2})-(2-\sqrt{2})}$

$= \dfrac{5-3\sqrt{2}}{1-\sqrt{2}}$

$= \dfrac{(5-3\sqrt{2})}{(1-\sqrt{2})} \times \dfrac{(1+\sqrt{2})}{(1+\sqrt{2})}$

$= \dfrac{5+5\sqrt{2}-3\sqrt{2}-6}{1+\sqrt{2}-\sqrt{2}-2}$

$= \dfrac{-1+2\sqrt{2}}{-1}$

$= 1-2\sqrt{2}\ (m=1, n=-2)$

9 a $x^2 - 10x + 16 = (x-8)(x-2)$

b Let $x = 8^y$

$8^{2y} - 10(8^y) + 16 = (8^y - 8)(8^y - 2) = 0$

So $8^y = 8$ or $8^y = 2$

$y = 1$ or $y = \frac{1}{3}$

10 a $x^2 - 8x = (x-4)^2 - 16$

Complete the square for $x^2 - 8x - 29$:

$x^2 - 8x - 29 = (x-4)^2 - 16 - 29$

$= (x-4)^2 - 45$

$(a = -4$ and $b = -45)$

b $x^2 - 8x - 29 = 0$

$(x-4)^2 - 45 = 0$

Use the result from part **a**:

$(x-4)^2 = 45$

Take the square root of both sides:

$x - 4 = \pm\sqrt{45}$

$x = 4 \pm \sqrt{45}$

$\sqrt{45} = \sqrt{9}\times\sqrt{5} = 3\sqrt{5}$

since $\sqrt{ab} = \sqrt{a}\sqrt{b}$

Roots are $4 \pm 3\sqrt{5}$

$(c = 4$ and $d = \pm 3)$

11 $f(a) = a(a-2)$ and $g(a) = a + 5$

$a(a-2) = a + 5$

$a^2 - 2a - a - 5 = 0$

$a^2 - 3a - 5 = 0$

Using the quadratic formula:

$a = \dfrac{3 \pm \sqrt{(-3)^2 - 4(1)(-5)}}{2(1)}$

$= \dfrac{3 \pm \sqrt{29}}{2}$

$= 4.19$ or -1.19

As $a > 0$, $a = 4.19$ (3 s.f.)

12 a $f(x) = x^2 - 6x + 18$

$x^2 - 6x = (x-3)^2 - 9$

Complete the square for $x^2 - 6x + 18$:

$x^2 - 6x + 18 = (x-3)^2 - 9 + 18$

$= (x-3)^2 + 9$

$a = 3$ and $b = 9$

b $y = x^2 - 6x + 18$

$y = (x-3)^2 + 9$

$(x-3)^2 \geq 0$

Squaring a number cannot give a negative result.

The minimum value of $(x - 3)^2$ is 0, when $x = 3$.

When $x = 3$, $y = 9$.

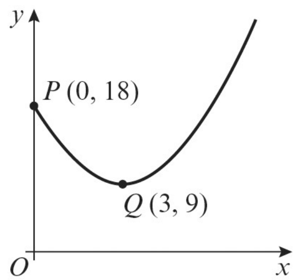
Q is the point (3, 9).

The curve crosses the y-axis where $x = 0$.

When $x = 0$, $y = 18$

P is the point (0, 18).

The graph of $y = x^2 - 6x + 18$ is a $\bigvee$ shape.



Use the information about P and Q to sketch the curve for $x \geq 0$; the part of the curve where $x < 0$ is not asked for.

12 c $y = (x-3)^2 + 9$

Put $y = 41$ into the equation of C.

$41 = (x-3)^2 + 9$

Subtract 9 from both sides.

$32 = (x-3)^2$

$(x-3)^2 = 32$

Take the square root of both sides.

$x - 3 = \pm\sqrt{32}$

$x = 3 \pm \sqrt{32}$

$= 3 \pm 4\sqrt{2}$

x-coordinate of R is $3 + 4\sqrt{2}$.

The other value is $3 - 4\sqrt{2}$ which is less than 0, so is not needed.

13 a Using the discriminant $b^2 - 4ac = 0$ for equal roots:

$(2\sqrt{2})^2 - 4(1)(k) = 0$

$8 - 4k = 0$

$k = 2$

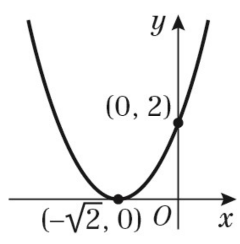
b $y = x^2 + 2\sqrt{2}x + 2$

$= (x + \sqrt{2})^2$

When $y = 0$, $(x + \sqrt{2})^2 = 0$,

so $x = -\sqrt{2}$

When $x = 0$, $y = 2$



14 a $g(x) = x^9 - 7x^6 - 8x^3$

$= x^3(x^6 - 7x^3 - 8)$

To factorise $x^6 - 7x^3 - 8$, let $y = x^3$

$y^2 - 7y - 8 = (y + 1)(y - 8)$

So $g(x) = x^3(x^3 + 1)(x^3 - 8)$

$a = 1$, $b = -8$

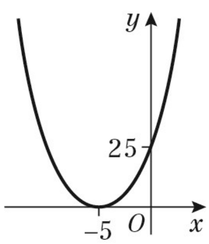
14 b $g(x) = x^3(x^3 + 1)(x^3 - 8) = 0$
$x^3 = 0, x^3 = -1$ or $x^3 = 8$
$x = 0, x = -1$ or $x = 2$

15 a $x^2 + 10x + 36$
$x^2 + 10x = 9\ (x+5)^2 - 25$
Complete the square for $x^2 + 10x + 36$:
$x^2 + 10x + 36 = (x-5)^2 - 25 + 36$
$= (x+5)^2 + 11$
$a = 5$ and $b = 11$

b $x^2 + 10x + 36 = 0$
$(x+5)^2 + 11 = 0$
'Hence' implied in part **a** must be used
$(x+5)^2 = -11$
A real number squared cannot be negative. There are no real roots.

c $x^2 + 10x + k = 0$
$a = 1, b = 10, c = k$
For equal roots, $b^2 = 4ac$
$10^2 = 4 \times 1 \times k$
$4k = 100$
$k = 25$

d $a = 1$, thus $a > 0$, so the graph of $x^2 + 10x + 25$ is a ∪ shape.
$x = 0$: $y = 0 + 0 + 25 = 25$
Meets y-axis at (0, 25).
$y = 0 : x^2 + 10x + 25 = 0$
$(x+5)(x+5) = 0$
$x = -5$
Meets x-axis at (−5, 0).



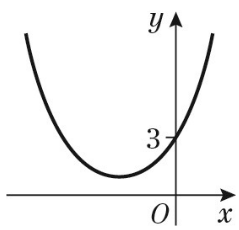
The graph meets the x-axis at just one point, so it 'touches' the x-axis.

16 a $x^2 + 2x + 3$
$x^2 + 2x = (x+1)^2 - 1$
Complete the square for $x^2 + 2x + 3$
$x^2 + 2x + 3 = (x+1)^2 - 1 + 3$
$= (x+1)^2 + 2$
$a = 1$ and $b = 2$

b $a = 1$, thus $a > 0$, so the graph of $y = x^2 + 2x + 3$ is a ∪ shape.
$x = 0 : y = 0 + 0 + 3$
Put $x = 0$ to find the intersection with the y-axis:
Meets y-axis at (0, 3).

Put $y = 0$ to find the intersection with the x-axis:
$y = 0 : x^2 + 2x + 3 = 0$
$(x + 1)^2 + 2 = 0$
$(x + 1)^2 = -2$

A real number squared cannot be negative, therefore, no real roots, so no intersection with the x-axis.



c $x^2 + 2x + 3$
$a = 1, b = 2, c = 3$
$b^2 - 4ac = 2^2 - 4 \times 1 \times 3$
$= -8$
Since the discriminant is negative, the equation has no real roots, so the graph does not cross the x-axis.

16 d $x^2 + kx + 3 = 0$

$a = 1, b = k, c = 3$

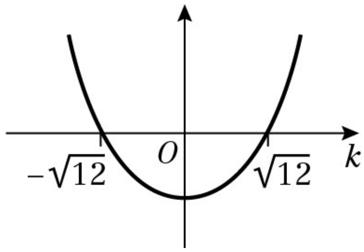
For no real roots, $b^2 < 4ac$

$k^2 < 12$

$k^2 - 12 < 0$

$\left(k+\sqrt{12}\right)\left(k-\sqrt{12}\right)<0$

This is a quadratic inequality with critical values $-\sqrt{12}$ and $\sqrt{12}$.



Critical values:

$k = -\sqrt{12}$, $k = \sqrt{12}$

$-\sqrt{12} < k < \sqrt{12}$

The surds can be simplified using $\sqrt{(ab)} = \sqrt{a}\sqrt{b}$

$\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

$\left(-2\sqrt{3} < k < 2\sqrt{3}\right)$

17 a $2x^2 - x(x-4) = 8$

$2x^2 - x^2 + 4x = 8$

$x^2 + 4x - 8 = 0$

17 b $x^2 + 4x - 8 = 0$

$x^2 + 4x = (x+2)^2 - 4$

$(x+2)^2 - 4 - 8 = 0$

$(x+2)^2 = 12$

$x + 2 = \pm\sqrt{12}$

$x = -2 \pm \sqrt{12}$

$\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$

$x = -2 \pm 2\sqrt{3}$

$a = -2$ and $b = 2$

Using $y = x - 4$:

$y = \left(-2 \pm 2\sqrt{3}\right) - 4$

$= -6 \pm 2\sqrt{3}$

Solution: $x = -2 \pm 2\sqrt{3}$

$y = -6 \pm 2\sqrt{3}$

18 a $3(2x + 1) > 5 - 2x$

$6x + 3 > 5 - 2x$

$6x + 2x + 3 > 5$

$8x > 2$

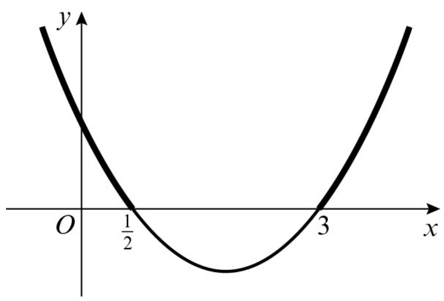
$x > \frac{1}{4}$

b $2x^2 - 7x + 3 = 0$

$(2x - 1)(x - 3) = 0$

$(2x - 1) = 0$ or $(x - 3) = 0$

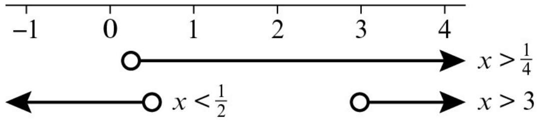
$x = \frac{1}{2}$ or $x = 3$



$2x^2 - 7x + 3 > 0$ where

$x < \frac{1}{2}$ or $x > 3$

18 c



$\frac{1}{4} < x < \frac{1}{2}$ or $x > 3$

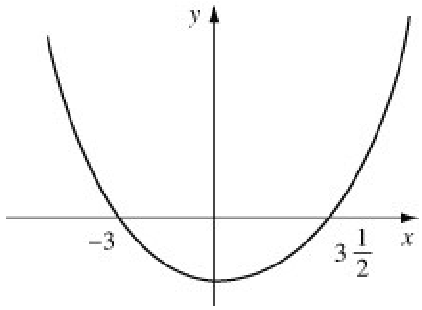
19 $-2(x+1) = x^2 - 5x + 2$
$-2x - 2 = x^2 - 5x + 2$
$x^2 - 3x + 4 = 0$
Using the discriminant
$b^2 - 4ac = (-3)^2 - 4(1)(4) = -7$
As $b^2 - 4ac < 0$, there are no real roots.
Hence there is no value of x for which $\text{p}(x) = \text{q}(x)$.

20 a $y = 5 - 2x$

$2x^2 - 3x - (5 - 2x) = 16$
$2x^2 - 3x - 5 + 2x = 16$
$2x^2 - x - 21 = 0$
$(2x - 7)(x + 3) = 0$
$x = 3\frac{1}{2},\ x = -3$
$x = 3\frac{1}{2}:\ y = 5 - 7 = -2$
$x = -3:\ y = 5 + 6 = 11$

Solution $x = 3\frac{1}{2},\ y = -2$
and $x = -3,\ y = 11$

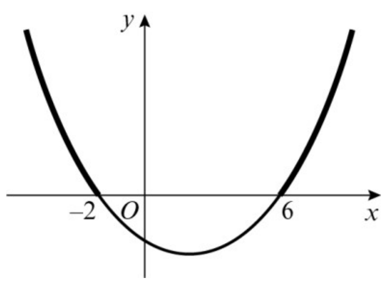
20 b The equations in part **a** could be written as $y = 5 - 2x$ and $y = 2x^2 - 3x - 16$.
Therefore, the solutions to $2x^2 - 3x - 16 = 5 - 2x$ are the same as for part **a**.
These are the critical values for $2x^2 - 3x - 16 > 5 - 2x$:
$x = 3\frac{1}{2}$ and $x = -3$.
$2x^2 - 3x - 16 > 5 - 2x$
$(2x^2 - 3x - 16 - 5 + 2x > 0)$
$2x^2 - x - 21 > 0$



$x < -3$ or $x > 3\frac{1}{2}$

21 a $x^2 + kx + (k+3) = 0$
$a = 1,\ b = k,\ c = k + 3$
$b^2 > 4ac$
$k^2 > 4(k+3)$
$k^2 > 4k + 12$
$k^2 - 4k - 12 > 0$

b $k^2 - 4k - 12 = 0$
$(k+2)(k-6) = 0$
$k = -2,\ k = 6$

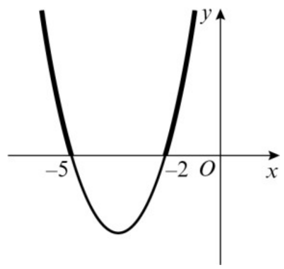


$k^2 - 4k - 12 > 0$ where $k < -2$ or $k > 6$

22 $\frac{6}{x+5} < 2$

Multiply both sides by $(x + 5)^2$
$6(x + 5) < 2(x + 5)^2$
$6x + 30 < 2x^2 + 20x + 50$
$2x^2 + 14x + 20 > 0$

Solve the quadratic to find the critical values.
$2x^2 + 14x + 20 = 0$
$2(x^2 + 7x + 10) = 0$
$2(x + 5)(x + 2) = 0$
$x = -5$ or $x = -2$



The solution is $x < -5$ or $x > -2$.

23 a $9 - x^2 = 0$
$(3 + x)(3 - x) = 0$
$x = -3$ or $x = 3$
When $x = 0, y = 9$

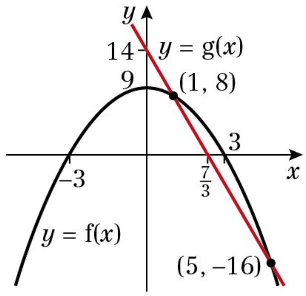
To work out the points of intersection, solve the equations simultaneously.
$9 - x^2 = 14 - 6x$
$x^2 - 6x + 5 = 0$
$(x - 5)(x - 1) = 0$
$x = 1$ or $x = 5$

When $x = 1, y = 8$
When $x = 5, y = -16$

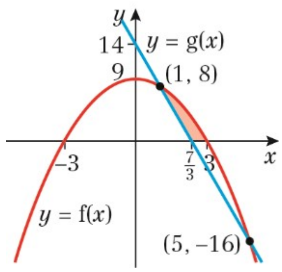
Let $14 - 6x = 0$
$x = \frac{14}{6} = \frac{7}{3}$

The line crosses the x-axis at $\left(\frac{7}{3}, 0\right)$.

23 a

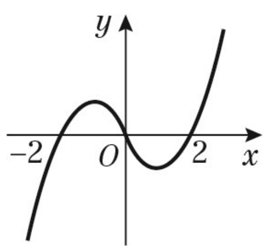


b



24 a $x^3 - 4x = x(x^2 - 4)$
$= x(x+2)(x-2)$

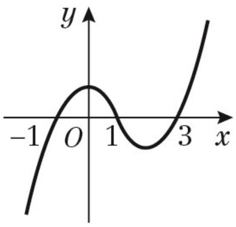
b Curve crosses the x-axis where $y = 0$
$x(x+2)(x-2) = 0$
$x = 0,\ x = -2,\ x = 2$
When $x = 0,\ y = 0$
When $x \to \infty,\ y \to \infty$
When $x \to -\infty,\ y \to -\infty$



Crosses the y-axis at (0, 0).
Crosses the x-axis at (−2, 0), (2, 0).

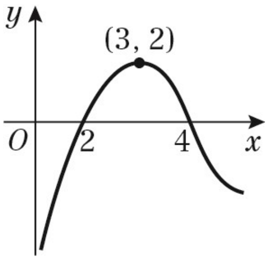
c $y = x^3 - 4x$
$y = (x-1)^3 - 4(x-1)$
This is a translation of +1 in the x-direction.

24 c



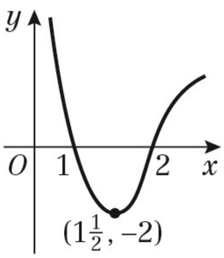
Crosses the x-axis at $(-1, 0)$, $(1, 0)$ and $(3, 0)$.

25 a



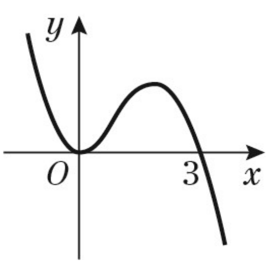
Crosses the x-axis at $(2, 0)$ and $(4, 0)$.
Image of P is $(3, 2)$.

b



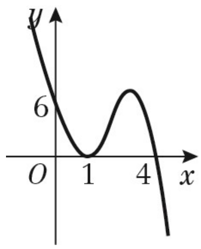
Crosses the x-axis at $(1, 0)$ and $(2, 0)$.
Image of P is $\left(1\frac{1}{2}, -2\right)$.

26 a



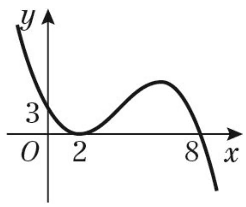
Meets the y-axis at $(0, 0)$.
Crosses the x-axis at $(3, 0)$.

26 b



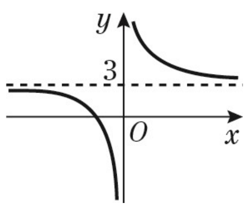
Crosses the y-axis at $(0, 6)$.
Meets the x-axis at $(1, 0)$ and crosses the x-axis at $(4, 0)$.

c



Crosses the y-axis at $(0, 3)$.
Meets the x-axis at $(2, 0)$ and crosses the x-axis at $(8, 0)$.

27 a



$y = 3$ is an asymptote.
$x = 0$ is an asymptote.

b The graph does not cross the y-axis (see sketch in part **a**).
Crosses the x-axis where $y = 0$:

$$\frac{1}{x} + 3 = 0$$

$$\frac{1}{x} = -3$$

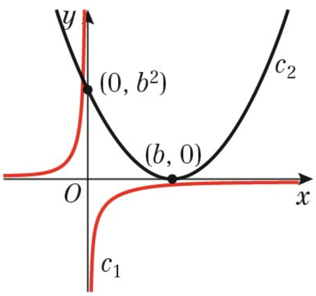
$$x = -\tfrac{1}{3}, \left(-\tfrac{1}{3}, 0\right)$$

28 a $y = -\text{f}(x)$ is a reflection in the x-axis of $y = \text{f}(x)$, so P is transformed to (6, 8).

b $y = \text{f}(x - 3)$ is a translation 3 units to the right of $y = \text{f}(x)$, so P is transformed to (9, −8).

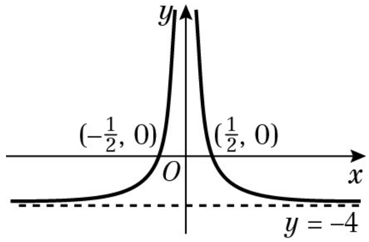
c $2y = \text{f}(x)$ is $y = \frac{1}{2}\text{f}(x)$ which is a vertical stretch scale factor $\frac{1}{2}$ of $y = \text{f}(x)$, so P is transformed to (6, −4).

29 a $y = -\frac{a}{x}$ is the curve $y = \frac{k}{x}$, $k < 0$
$y = (x - b)^2$ is a translation, b units to the right of the curve $y = x^2$
When $x = 0$, $y = b^2$
When $y = 0$, $x = b$



b The graphs have 1 point of intersection.

30 a $y = \frac{1}{x^2} - 4$ is a translation $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ of $y = \frac{1}{x^2}$



30 b When $y = \frac{1}{(x+k)^2} - 4$ passes through the origin, $x = 0$ and y = 0.

So $\frac{1}{k^2} - 4 = 0$

$$\frac{1}{k^2} = 4$$

$$k = \pm\frac{1}{2}$$

Challenge

1 a $x^2 - 10x + 9 = 0$
$(x - 1)(x - 9) = 0$
$x = 1$ or $x = 9$

b $3^{x-2}(3^x - 10) = -1$
$3^{2x-2} - 10 \times 3^{x-2} + 1 = 0$
Multiply by 3^2:
$3^{2x} - 10 \times 3^x + 9 = 0$
Let $y = 3^x$
$y^2 - 10y + 9 = 0$
Using your answers from part **a**
$y = 1$ or 9
$3^x = 1$ or $3^x = 9$
$x = 0$ or $x = 2$

2 Let x and y be the length and width of the rectangle respectively.
Area $= xy = 6$
Perimeter $= 2x + 2y = 8\sqrt{2}$
$2y = 8\sqrt{2} - 2x$
$y = 4\sqrt{2} - x$
Solving simultaneously:
$x\left(4\sqrt{2} - x\right) = 6$
$x^2 - 4\sqrt{2}x + 6 = 0$

Using the quadratic formula:

$$x = \frac{4\sqrt{2} \pm \sqrt{(4\sqrt{2})^2 - 4(1)(6)}}{2(1)}$$
$$= \frac{4\sqrt{2} \pm \sqrt{8}}{2}$$
$$= \frac{4\sqrt{2} \pm 2\sqrt{2}}{2}$$

$x = \sqrt{2}$ or $x = 3\sqrt{2}$

When $x = \sqrt{2}$, $y = 3\sqrt{2}$
When $x = 3\sqrt{2}$, $y = \sqrt{2}$

The dimensions of the rectangle are $\sqrt{2}$ cm and $3\sqrt{2}$ cm.

3 Solving simultaneously
$3x^3 + x^2 - x = 2x(x - 1)(x + 1)$
$3x^3 + x^2 - x = 2x(x^2 - 1)$
$3x^3 + x^2 - x = 2x^3 - 2x$
$x^3 + x^2 + x = 0$
$x(x^2 + x + 1) = 0$

The discriminant of $x^2 + x + 1$
$b^2 - 4ac = 1^2 - 4(1)(1) = -3$.
$-3 < 0$, so there are no real solutions for $x^2 + x + 1$.

The only solution is (0, 0).